

# YET ANOTHER PROOF OF THE ADO THEOREM

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**ABSTRACT.** We give a simple proof of the Birkhoff theorem about existence of a faithful representation for any finite-dimensional nilpotent Lie algebra of characteristic zero.

## 1. INTRODUCTION

The Ado theorem says that any finite-dimensional Lie algebra admits a faithful finite-dimensional representation. It was first proved by Ado in 1935 [A]<sup>†</sup> using a Lie-group technique (and as such, was restricted to the fields of complex and real numbers). Since then a few different proofs were given, including purely algebraic ones and those valid in the positive characteristic (the latter is due to Iwasawa; cf. [Bo, §7, Exercice 5]). A relatively recent new proof, due to Neretin, is given in [N]. All the known proofs involve the universal enveloping algebra, an infinite-dimensional object, and nontrivial facts about it (notably, the Poincaré–Birkhoff–Witt theorem). Moreover, the arguments are different in zero and positive characteristics. Somewhat unusually, the positive characteristic case of the theorem is much easier than the characteristic zero one, due to the possibility to employ finite-dimensional induced modules constructed with the aid of a reduced finite-dimensional version of the universal enveloping algebra (in characteristic zero, such modules would be infinite). Perhaps because of all this, the Ado theorem is sometimes referred – in writings, and also in talks and private conversations – as a “strange theorem” ([N]) “surprisingly tricky to prove” ([T, §2.3]).

Here we give an entirely different proof of the Ado theorem, basing on properties of free nilpotent Lie algebras, and on simple combinatorics related to the tensor product of representations. The proof is elementary and does not involve universal enveloping algebras (in fact, it does not involve associative algebras at all and is intrinsic to the category of finite-dimensional Lie algebras). Another interesting feature of the proof is that it employs induction on the dimension of the algebra – not “from below”, as in many existing proofs of the Ado theorem, but “from above”, descending from an algebra for which the Ado theorem is already established.

The drawbacks of the proof are that it is valid for nilpotent algebras and in characteristic zero only (first established in full generality by Birkhoff in 1937 [Bi], thus sometimes referred as “the Birkhoff theorem”).

We present the proof in Section 2 as a series of (elementary) lemmas. The Ado theorem is one of the cornerstone results of today’s structure theory of Lie algebras, used in plenty of proofs and arguments. When pretending to give a new proof of such a basic result, one should be especially careful not to fall into the trap of circular arguments. That is why, even when using known and/or elementary results, we at least outline their proofs. We also carefully isolate places where we need assumptions such as nilpotency of the algebra and characteristic zero of the ground field.

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<sup>†</sup> A bit of trivia: Sophus Lie had no doubt that (speaking in modern terms) every finite-dimensional Lie algebra admits a faithful finite-dimensional representation, but he was unable to prove this (cf. [LE, footnote at p. 598]). Nikolai Grigorievich Chebotarev has put his student Igor Dmitrievich Ado to the task. Ado presented his work as a Candidate (a Russian equivalent of PhD) dissertation, but was awarded a Doctor degree (a Russian equivalent of Habilitation) instead, an extremely rare event in Russian academic officialdom.

Being different, our proof still shares with all the previous proofs some elementary (and seemingly unavoidable in this context) tricks and observations. In particular, as for a centerless Lie algebra the Ado theorem is trivial (the adjoint representation will do), we will concentrate on central elements, and take the direct sum of representations to assemble “local” Ado properties (nonvanishing on particular elements) into the “global” one (faithfulness).

In Section 3 we speculate about possibility to extend this proof to the case of positive characteristic.

## 2. THE PROOF

In what follows, the ground field  $K$  is arbitrary, and all algebras, modules and vector spaces are finite-dimensional, unless stated otherwise.

A Lie algebra structure on the tensor product  $L \otimes A$  of a Lie algebra  $L$  and an associative commutative algebra  $A$  is defined by the obvious factor-wise multiplication:  $[x \otimes a, y \otimes b] = [x, y] \otimes ab$  for  $x, y \in L, a, b \in A$  (such Lie algebras are dubbed as *current Lie algebras*, the term coming from physics where they play a role).

**Lemma 2.1** (EMBEDDING OF GRADED ALGEBRAS INTO A TENSOR PRODUCT). *An  $\mathbb{N}$ -graded Lie algebra  $L$  is embedded into the Lie algebra  $L \otimes tK[t]/(t^n)$  for some  $n \in \mathbb{N}$ .*

(We denote by  $\mathbb{N}$  the set of all positive integers, so the algebras we consider are positively-graded).

*Proof.* Let  $L = \bigoplus_{i=1}^{n-1} L_i$  be the  $\mathbb{N}$ -grading (necessarily finite, as  $L$  is finite-dimensional). The algebra  $L$  is embedded into  $L \otimes tK[t]$  via  $x \mapsto x \otimes t^i$ , where  $x \in L_i, 1 \leq i \leq n-1$ . Since  $[L_i \otimes t^i, L_j \otimes t^j] = 0$  if  $i + j \geq n$ , this embedding factors through the ideal  $L \otimes t^n K[t]$ .  $\square$

**Remark 2.2.** The statement is obviously true for arbitrary, not necessary Lie,  $\mathbb{N}$ -graded algebras.

For the purpose of this note, a representation  $\rho$  of a Lie algebra  $L$  (or the corresponding  $L$ -module) will be called *nilpotent*, if  $\rho(x)$  is a nilpotent linear map for any  $x \in L$  (of course, due to the Engel theorem, this is equivalent to existence of a positive integer  $n$  such that  $\rho(x_1) \cdots \rho(x_n) = 0$  for any  $x_1, \dots, x_n \in L$ , but we will not need that).

Recall that given a representation  $\rho : L \rightarrow \text{End}(V)$  of a Lie algebra  $L$ , an *1-cocycle* is a linear map  $\varphi : L \rightarrow V$  such that

$$\varphi([x, y]) - \rho(x)(\varphi(y)) + \rho(y)(\varphi(x)) = 0$$

for any  $x, y \in L$ . The space of all such 1-cocycles is denoted by  $Z^1(L, V)$ , and they may be thought as derivations of  $L$  with values in the  $L$ -module  $V$ .

**Lemma 2.3** (A NONDEGENERATE COCYCLE IMPLIES ADO). *Let  $L$  be a Lie algebra,  $V$  an  $L$ -module (respectively, nilpotent  $L$ -module), and  $\varphi$  is 1-cocycle in  $Z^1(L, V)$  such that  $\text{Ker } \varphi = 0$ . Then  $L$  has a faithful representation (respectively, faithful nilpotent representation).*

*Proof.* The required representation  $\rho$  is given by an action of  $L$  on  $V \oplus Z^1(L, V)$  (direct sum of vector spaces), defined naturally on the first direct summand, and via  $\rho(x)(\psi) = \psi(x)$  for  $x \in L$  and  $\psi \in Z^1(L, V)$ , on the second direct summand.  $\square$

**Remark 2.4.** It is possible to extend this statement, via induction, to higher-order cocycles, but we will not need this (but see Remark 2.12 below).

**Lemma 2.5** (ADO FOR GRADED ALGEBRAS). *An  $\mathbb{N}$ -graded Lie algebra over a field of characteristic zero has a faithful nilpotent representation.*

*Proof.* Let  $L$  be such Lie algebra. By Lemma 2.1,  $L$  is embedded into  $L \otimes tK[t]/(t^n)$ . The derivation (i.e., 1-cocycle with values in the adjoint module)  $\text{id}_L \otimes t \frac{d}{dt}$  of  $L \otimes tK[t]/(t^n)$  acts on a nonzero element  $\sum_{i=1}^{n-1} x_i \otimes t^i$  non-trivially, hence has zero kernel. By Lemma 2.3,  $L \otimes tK[t]/(t^n)$  has a faithful nilpotent representation. Hence so does its subalgebra  $L$ .  $\square$

**Lemma 2.6** (ADO FOR FREE NILPOTENT ALGEBRAS). *A free nilpotent Lie algebra of finite rank over a field of characteristic zero has a faithful nilpotent representation.*

*Proof.* A free nilpotent Lie algebra of finite rank is finite-dimensional and  $\mathbb{N}$ -graded. Apply Lemma 2.5.  $\square$

**Lemma 2.7** (LOCAL ADO IMPLIES GLOBAL ADO<sup>†</sup>). *Let  $L$  be a Lie algebra such that for any nonzero  $x \in L$  there is a nilpotent representation  $\rho_x$  of  $L$  such that  $\rho_x(x) \neq 0$ . Then  $L$  has a faithful nilpotent representation.*

*Proof.* Pick a nonzero element  $x_1 \in L$ , and set  $\rho_1 = \rho_{x_1}$ . If  $\rho_1$  is faithful, we are done, if not, there is a nonzero  $x_2 \in L$  such that  $\rho_1(x_2) = 0$ . Set  $\rho_2 = \rho_1 \oplus \rho_{x_2}$ . Note that

$$\text{Ker } \rho_2 = \text{Ker } \rho_1 \cap \text{Ker } \rho_{x_2} \subset \text{Ker } \rho_1,$$

and the inclusion is strict, since  $x_2$  does not belong to the left-hand side, but belongs to the right-hand side. Repeating this process, we get a series of representations  $\rho_1, \rho_2, \dots$ , with strictly decreasing kernels. Since  $L$  is finite-dimensional, this process will terminate in a finite number of steps on a faithful representation  $\rho$ . Being the direct sum of nilpotent representations,  $\rho$  is also nilpotent.  $\square$

**Lemma 2.8** (ABUNDANCE OF IDEALS OF CODIMENSION 1). *Let  $I$  be a nonzero ideal of a nilpotent Lie algebra  $L$ . Then  $I$  contains an ideal  $J$  of  $L$  of codimension 1 in  $I$  such that  $[L, I] \subseteq J$ .*

*Proof.* This elementary result follows almost immediately from the definition of a nilpotent Lie algebra by a rudimentary Jordan–Hölder-like argument. Namely, there is a chain of ideals of  $L$

$$0 = I_0 \subset I_1 \subset \dots \subset I_{n-1} \subset I_n = L$$

such that  $\dim I_i/I_{i-1} = 1$  and  $[L, I_i] \subseteq I_{i-1}$  for any  $i = 1, 2, \dots, n$  (cf., e.g., [Bo, §4, Proposition 1]). The set  $\{i \mid I \subseteq I_i\}$  contains  $n$  (and hence is nonempty), and does not contain 0 (since  $I$  is nonzero). Let  $k$  be the minimal element in this set. Then

$$0 < \dim I/(I \cap I_{k-1}) = \dim(I + I_{k-1})/I_{k-1} \leq \dim I_k/I_{k-1} = 1,$$

and hence the ideal  $I \cap I_{k-1}$  of  $L$  is of codimension 1 in  $I$ . We also have

$$[L, I] \subseteq I \cap [L, I_k] \subseteq I \cap I_{k-1},$$

as required.  $\square$

**Lemma 2.9** (FACTORIZATION OF LINEAR MAPS). *Let  $V$  be a vector space,  $f, g$  linear maps in  $\text{End}(V)$ , and  $\text{Ker } f \subseteq \text{Ker } g$ . Then there is a linear map  $h$  in  $\text{End}(V)$  such that  $g = h \circ f$ .*

*Proof.* This is an elementary linear algebra (cf., e.g., [KM, Proposition 6.8]). Fix a basis  $e_1, \dots, e_n$  in  $V$ , and define a linear map  $h : \text{Im } f \rightarrow V$  by sending  $f(e_i)$  to  $g(e_i)$ ,  $i = 1, \dots, n$ . This map is well-defined, since if  $\sum_i \lambda_i f(e_i) = \sum_i \mu_i f(e_i)$  for some  $\lambda_i, \mu_i \in K$ , then  $\sum_i (\lambda_i - \mu_i) e_i$  lies in  $\text{Ker } f$  and hence in  $\text{Ker } g$ , and, consequently,  $h(\sum_i \lambda_i f(e_i)) = h(\sum_i \mu_i f(e_i))$ . Extend  $h$  to the whole  $V$  arbitrarily, say, by mapping a subspace complementary to  $\text{Im } f$  to zero.  $\square$

The next lemma contains the core arguments, of combinatorial character.

**Lemma 2.10** (DISTINGUISHING ELEMENTS BY REPRESENTATION KERNELS). *Let  $L$  be a Lie algebra over a field of characteristic zero, having a faithful nilpotent representation. Then for any two linearly independent elements  $x, y \in L$ , there is a nilpotent representation  $\rho$  of  $L$  such that  $\text{Ker } \rho(x) \not\subseteq \text{Ker } \rho(y)$ .*

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<sup>†</sup> A reader of an earlier draft of this note remarked that after the present array of lemmas, one might expect the next one to be called MUCH ADO ABOUT NOTHING. The lemmas are elementary indeed.

*Proof.* Suppose the contrary: there are two linearly independent elements  $x, y \in L$  such that for any nilpotent representation  $\rho$  of  $L$ ,  $\text{Ker } \rho(x) \subseteq \text{Ker } \rho(y)$ . By Lemma 2.9,

$$(1) \quad \rho(y) = h_\rho \circ \rho(x)$$

for some linear map  $h_\rho$ .

Let  $\rho : L \rightarrow \text{End}(V)$  and  $\tau : L \rightarrow \text{End}(W)$  be two nilpotent representations of  $L$ , and let  $n$  and  $m$  be indices of nilpotency of  $\rho(x)$  and  $\tau(x)$ , respectively. The tensor product  $\rho \otimes \tau$  is also nilpotent, so writing the condition (1) for  $\rho \otimes \tau$  and for vectors  $\rho(x)^{n-2}(v) \otimes \tau(x)^{m-1}(w)$  and  $\rho(x)^{n-1}(v) \otimes \tau(x)^{m-2}(w)$ , and taking into account the same condition for  $\rho$  and  $\tau$ , we get respectively

$$h_\rho(\rho(x)^{n-1}(v)) \otimes \tau(x)^{m-1}(w) = h_{\rho \otimes \tau}(\rho(x)^{n-1}(v) \otimes \tau(x)^{m-1}(w))$$

and

$$\rho(x)^{n-1}(v) \otimes h_\tau(\tau(x)^{m-1}(w)) = h_{\rho \otimes \tau}(\rho(x)^{n-1}(v) \otimes \tau(x)^{m-1}(w))$$

for any  $v \in V$ ,  $w \in W$ . This implies that the linear maps  $h_\rho \otimes \text{id}$  and  $\text{id} \otimes h_\tau$  coincide on the vector space  $\rho(x)^{n-1}(V) \otimes \tau(x)^{m-1}(W)$ , whence

$$(2) \quad h_\rho(\rho(x)^{n-1}(v)) = \lambda \rho(x)^{n-1}(v)$$

and

$$h_\tau(\tau(x)^{m-1}(w)) = \lambda \tau(x)^{m-1}(w)$$

for some  $\lambda \in K$ . Since this holds for any pair of representations  $\rho, \tau$ , (2) holds for any nilpotent representation  $\rho$  of  $L$  for some uniform value  $\lambda \in K$ .

Further, writing the condition (1) for the tensor square  $\rho \otimes \tau$  for vectors  $\rho(x)^{n-3}(v) \otimes \tau(x)^{m-1}(w)$ ,  $\rho(x)^{n-1}(v) \otimes \tau(x)^{m-3}(w)$ , and  $\rho(x)^{n-2}(v) \otimes \tau(x)^{m-2}(w)$ , and taking into account (2), we get respectively

$$\begin{aligned} h_\rho(\rho(x)^{n-2}(v)) \otimes \tau(x)^{m-1}(w) &= h_{\rho \otimes \tau}(\rho(x)^{n-2}(v) \otimes \tau(x)^{m-1}(w)), \\ \rho(x)^{n-1}(v) \otimes h_\tau(\tau(x)^{m-2}(w)) &= h_{\rho \otimes \tau}(\rho(x)^{n-1}(v) \otimes \tau(x)^{m-2}(w)), \end{aligned}$$

and

$$\begin{aligned} (3) \quad \lambda(\rho(x)^{n-1}(v) \otimes \tau(x)^{m-2}(w) + \rho(x)^{n-2}(v) \otimes \tau(x)^{m-1}(w)) \\ = h_{\rho \otimes \tau}(\rho(x)^{n-1}(v) \otimes \tau(x)^{m-2}(w) + \rho(x)^{n-2}(v) \otimes \tau(x)^{m-1}(w)). \end{aligned}$$

Summing up the first two of these equalities, and subtracting the third one, we get that the linear maps  $(h_\rho - \lambda \text{id}) \otimes \tau(x)$  and  $\rho(x) \otimes (h_\tau - \lambda \text{id})$  coincide on the vector space  $\rho(x)^{n-2}(V) \otimes \tau(x)^{m-2}(W)$ , whence  $h_\rho - \lambda \text{id} = -\mu \rho(x)$  and  $h_\tau - \lambda \text{id} = -\mu \tau(x)$  for some  $\mu \in K$  as linear maps on  $\rho(x)^{n-2}(V)$  and  $\tau(x)^{m-2}(W)$ , respectively. As this holds for any pair of nilpotent representations  $\rho, \tau$ , we get that

$$(4) \quad h_\rho(\rho(x)^{n-2}(v)) = \lambda \rho(x)^{n-2}(v) + \mu \rho(x)^{n-1}(v)$$

for any nilpotent representation  $\rho$  of  $L$ , for some uniform value  $\mu \in K$ .

On the other hand, the index of nilpotency of  $(\rho \otimes \rho)(x)$  is equal to  $2n - 1$ , so writing (4) for the tensor square  $\rho \otimes \rho$ , and taking into account (3) in the case  $\tau = \rho$  (and  $m = n$ ), we get

$$(5) \quad \binom{2n-2}{n-1} \mu \rho(x)^{n-1}(v) \otimes \rho(x)^{n-1}(w) = 0$$

for any  $v, w \in V$ , what implies  $\mu = 0$ , and, according to (4),

$$h_\rho(\rho(x)^{n-2}(v)) = \lambda \rho(x)^{n-2}(v)$$

for any nilpotent representation  $\rho$  of  $L$ .

Repeating this procedure (considering on each step the condition (1) for the tensor square of two representations  $\rho, \tau$ , and for all vectors of the form  $\rho(x)^i(v) \otimes \tau(x)^j(w)$  with  $i + j$  equal to the index of nilpotency of  $\rho \otimes \tau$  minus  $(k + 2)$ ), we consecutively arrive at the equalities

$$h_\rho(\rho(x)^k(v)) = \lambda \rho(x)^k(v)$$

for any  $1 \leq k \leq n$ . For  $k = 1$  this means that  $h_\rho = \lambda \text{id}$ , and hence  $\rho(y - \lambda x)$  vanishes for any nilpotent representation  $\rho$  of  $L$ , whence  $y - \lambda x = 0$ , a contradiction.  $\square$

Finally, we can glue all of this together:

**Theorem 2.11** (ADO FOR NILPOTENT ALGEBRAS). *A nilpotent Lie algebra over a field of characteristic zero has a faithful nilpotent representation.*

*Proof.* Present a nilpotent Lie algebra  $L$  as a quotient of a free nilpotent Lie algebra  $F$  of a finite rank:  $L = F/I$ . We will proceed by induction on the dimension of  $I$ . The case  $I = 0$  is covered by Lemma 2.6.

Suppose that  $I$  is nonzero. By Lemma 2.8, there is an ideal  $J$  of  $F$  such that  $J \subset I$ ,  $\dim I/J = 1$ , and  $[F, I] \subseteq J$ . Consequently,  $S = F/J$  is an extension of  $L$  by an one-dimensional central ideal, say  $K\tilde{z}$ .

Take an arbitrary nonzero  $x \in L$  and consider its preimage  $\tilde{x}$  in  $S$ . By the induction assumption,  $S$  has a faithful nilpotent representation, and by Lemma 2.10, there is a nilpotent representation  $\rho : S \rightarrow \text{End}(V)$  such that

$$(6) \quad \text{Ker } \rho(\tilde{z}) \not\subseteq \text{Ker } \rho(\tilde{x}).$$

Since  $\tilde{z}$  lies in the center of  $S$ ,  $\rho(\tilde{z})$  commutes with all maps from  $\rho(S)$ , and hence the space  $\text{Ker } \rho(\tilde{z})$  is an  $S$ -submodule of  $V$ , which also carries a natural structure of a (nilpotent)  $L$ -module, on which  $x$  acts nontrivially, due to (6). By Lemma 2.7,  $L$  has a faithful nilpotent representation.  $\square$

**Remark 2.12.** In the proof above, we may try to take a dual route, utilizing images instead of kernels. Namely, for any representation  $\rho : S \rightarrow \text{End}(V)$ , the space  $V/\text{Im } \rho(\tilde{z})$  also carries a natural structure of an  $L$ -module. If  $\text{Im } \rho(\tilde{x}) \not\subseteq \text{Im } \rho(\tilde{z})$ , then  $x$  acts on the latter module nontrivially, and this action is nilpotent if  $\rho$  is nilpotent, so we may assume that  $\text{Im } \rho(\tilde{x}) \subseteq \text{Im } \rho(\tilde{z})$  for any nilpotent representation  $\rho$  of  $S$ . By the statement dual to Lemma 2.9, the latter condition may be rewritten as a dual one to (1):

$$\rho(\tilde{x}) = \rho(\tilde{z}) \circ h_\rho$$

for some linear map  $h_\rho : V \rightarrow V$ . Additionally, we may be helped by the fact that the element  $\tilde{x}$  in these considerations may be assumed to be central. Indeed, write the central extension  $S$  of  $L$  as the vector space direct sum  $S = L \oplus K\tilde{z}$ , with multiplication

$$\{u, v\} = [u, v] + \varphi(u, v)\tilde{z}$$

for  $u, v \in L$ , where  $[\cdot, \cdot]$  is multiplication in  $L$ , and  $\varphi$  is a 2-cocycle on  $L$  with values in  $K$ . Then we may apply a “local” version of Lemma 2.3 for 2-cocycles to deduce that  $\varphi(x, L) = 0$ , i.e.  $\tilde{x}$  is central in  $S$ . But even with this additional input, to prove the appropriate dual version of Lemma 2.10 seems to be more tricky, and requires an extensive consideration of associative envelopes of  $\rho(S)$ ’s in the appropriate matrix algebra.

### 3. AN ALTERNATIVE ROUTE

In the proof of the Ado theorem for nilpotent algebras in Section 2, the characteristic zero of the ground field  $K$  is needed in two places: first, in the proof of Lemma 2.5, to ensure that there is a derivation  $D$  of  $tK[t]/(t^n)$  such that  $\sum_{i \geq 1} x_i \otimes D(t^i) \neq 0$  (what is wrong if the characteristic of  $K$  is  $p > 0$ , and all the exponents  $i$  in nonzero terms of  $\sum_{i \geq 1} x_i \otimes t^i$  divide  $p$ ); and second, in the proof of Lemma 2.10, to ensure that the binomial coefficients arising in binomial-like formulas

for powers of Lie algebra elements actions on tensor products of representations, like those in (5), do not vanish.

Here we outline a possible alternative approach which should include the case of positive characteristic. In Section 2 we apply an elementary embedding of Lemma 2.1 to free nilpotent Lie algebras of finite rank. A little bit more involved argument establishes a similar embedding for a broader class algebras. This argument is sometimes phrased in terms of “generic elements” or “generic matrices” and is often employed in the theory of varieties of Lie, associative, and other kinds of algebras. Namely, a relatively free algebra  $F$  of no more than countable rank in a variety generated by a finite-dimensional Lie algebra  $L$  is embedded into

$$L \otimes \left( t_1 K[t_1, \dots, t_k] + \dots + t_k K[t_1, \dots, t_k] \right)$$

for some  $k \in \mathbb{N}$  (cf., e.g., [R, Chapter I, Lemma 5.1] or [Za, Proof of Proposition 1.3]). In the case of nilpotent  $L$  one can do even better and embed  $F$  into  $L \otimes tK[t]$  (cf. [Za, Lemma 1.1]). Also, in the case of nilpotent  $L$  this embedding obviously factors through the ideal  $L \otimes t^n K[t]$ , where  $n$  is the index of nilpotency of  $L$ , similarly as in Lemma 2.1.

(Note parenthetically that this or similar arguments are often coupled with the Ado theorem to establish an embedding of a relatively free algebra in some variety of Lie algebras into an algebra with that or another finiteness condition (cf., e.g., [Za]). Here we reverse this line of reasonings and use this argument to outline a possible route to the Ado theorem).

As the adjoint representation is nonzero on non-central elements, in view of Lemma 2.7, in order to establish the Ado theorem for  $F$ , it will be enough to prove that for any nonzero central element of  $F$ , there is a representation not vanishing on that element. By an easy inductive argument, one may reduce considerations to the case where the center  $Z(L)$  of  $L$  is one-dimensional, so to cover the characteristic  $p$  case in the proof of Lemma 2.5, it will be enough to consider elements of  $F$  of the form

$$z \otimes (\lambda_1 t^p + \lambda_2 t^{2p} + \dots),$$

where  $z \in Z(L)$  and  $\lambda_i \in K$ , which probably could be dealt with using additional considerations based on relative freeness of  $F$ .

Additionally, one may try to employ derivations of  $L \otimes tK[t]/(t^n)$  of the form other than  $\text{id}_L \otimes D$ , where  $D$  is a derivation of  $tK[t]/(t^n)$ . As explained in [Zu, §3], the full description of derivations of such current Lie algebras is, probably, a difficult task, but one may try, for example, to employ derivations of the forms listed in [Zu, Theorem 3].

After establishing the Ado theorem for any relatively free algebra  $F$  in a variety generated by a nilpotent Lie algebra  $L$ , we may proceed the same way as in the proof of Theorem in Section 2, by induction on the dimension of ideal of relations determining  $L$ . To establish Lemma 2.10 in characteristic  $p$ , an additional care will be needed to deal with vanishing of binomial coefficients occurring in the proof.

All this, however, will result in a quite long and involved proof – at least much more long and involved than the existing proof of the full-fledged Ado theorem in positive characteristic – so we will not pursue this approach.

A final remark: in all existing proofs of the Ado theorem in characteristic zero, the general case is derived from the nilpotent one. In this respect, §2 is a good start. However, all such derivations employ the universal enveloping algebra in a non-trivial way. A short, “natural”, and characteristic-free proof of the Ado theorem is yet to be found.

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